

# Chaos and Brane-worlds<sup>†</sup>

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**Abstract.** The early time behaviour of brane-world models is analysed in the presence of anisotropic stresses. It is shown that the initial singularity cannot be isotropic, unless there is also an isotropic fluid stiffer than radiation present. Also, a magnetic Bianchi type I brane-world is analysed in detail. It is known that the Einstein equations for the magnetic Bianchi type I models are in general oscillatory and are believed to be chaotic, but in the brane-world model this chaotic behaviour does not seem to be possible.

**Keywords:** brane-worlds, magnetic fields, chaos, singularity

“My brain is always chaotic.”

## 1. Introduction

The classical Einstein equations have been shown to have a very peculiar feature, namely chaos [1–8]. As the initial singularity is approached some cosmological solutions seem to oscillate chaotically. This type of behaviour in the general relativistic vacuum Bianchi type VIII and IX models have been well studied [9], and chaotic behaviour for the general relativistic magnetic Bianchi type I model has been conjectured [10].

The question we will address here is the following: Is the same chaotic behaviour present in the relatively newly proposed brane-world models? [11, 12]. Earlier investigations have shown that the brane-world models are more sensitive to the matter content than its general relativistic (GR) counterpart, especially at high energies [13]. The reason for this is that the Friedmann equation on the brane contains quadratic terms in the energy-density compared to the usual linear term in ordinary GR cosmology.

An important set of solutions in the study of the classical general relativistic cosmologies are the ones found by Kasner [14]. These are vacuum Bianchi type I solutions with metric

$$ds^2 = -dt^2 + t^{2p_1}dx^2 + t^{2p_2}dy^2 + t^{2p_3}dz^2 \quad (1)$$

where  $\sum_i p_i = \sum_i p_i^2 = 1$ . They correspond to highly anisotropic universes and have a very special role in GR cosmology. For most types

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<sup>†</sup> Based on a work done in collaboration with J.D. Barrow.



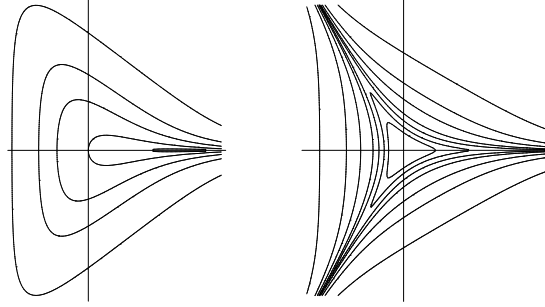


Figure 1. The potentials in the Bianchi type VIII (left) and type XI model (right).

of matter (except, for example, a stiff fluid), all spatially homogeneous universe models will – as we approach the initial singularity

- asymptote to one of the Kasner solutions, *or*
- have **chaotic** oscillations between different Kasner epochs.

(c.f. [15, 16]) In particular, in the latter case the Kasner solutions are unstable in the past while in the former they are stable. Hence, in GR cosmology the initial singularity will in general be anisotropic.

The chaos can be seen more intuitively from a Hamiltonian point of view. In this picture the Bianchi type VIII and IX models have a potential of a triangular shape, depicted in Fig. 1. The “universe point” will bounce off the exponentially steep walls and cause the universe to move from one Kasner epoch into another. This oscillation is chaotic and is an attractor set for most matter configurations.

## 2. Chaos in the classical magnetic Bianchi type I

The presence of cosmic magnetic fields tend to mimic the behaviour in the vacuum type VIII and IX models. In the type VIII and IX cases the three-curvature gives rise to the walls seen in Fig. 1. The magnetic fields give rise to similar walls and a similar chaotic behaviour results.

In particular, using the dynamical systems approach, LeBlanc [10] investigated the Bianchi type I with a magnetic field and a perfect fluid. The system has no equilibrium points which act as past attractors, but there exists a compact subset of phase space which generates chaotic oscillations on the Kasner circle. This set was conjectured to be an attractor into the past for generic models. There exist more complicated models having chaotic behaviour, but due to the simpleness of the magnetic Bianchi type I model it can serve as an effective and simple model to study this type of behaviour.

### 3. Brane-worlds

Based on investigations of brane-worlds with isotropic fluids it has been suggested by some authors [17–21] that brane-worlds have an isotropic singularity<sup>1</sup> and thus have no chaotic behaviour. However, this is an oversimplification of the early time behaviour of brane-worlds. Anisotropic stresses are inevitable in anisotropic spacetimes at early times because of the presence of collisionless gravitons, collisionless asymptotically-free particles, and electric and magnetic fields. These anisotropic stresses can arise because of intrinsic anisotropic stresses on the brane or via the induced graviton stresses from the bulk. Here we will consider the simplest example: a pure magnetic field on a flat anisotropic brane with Bianchi type I geometry.

The evolution equations on the brane are as follows[26, 27]. The Friedmann equation,

$$H^2 = \frac{\Lambda}{3} + \frac{1}{6}\sigma^{\mu\nu}\sigma_{\mu\nu} - \frac{1}{6}{}^{(3)}\mathcal{R} + \frac{\kappa^2}{3}\rho + \frac{\kappa^2}{6\lambda}\left[\rho^2 - \frac{3}{2}\pi_{\mu\nu}\pi^{\mu\nu}\right] + \frac{2\mathcal{U}}{\kappa^2\lambda}, \quad (2)$$

the shear propagation equations,

$$\dot{\sigma}_{\langle\mu\nu\rangle} + \Theta\sigma_{\mu\nu} = \kappa^2\pi_{\mu\nu} - {}^{(3)}\mathcal{R}_{\langle\mu\nu\rangle} + \frac{\kappa^2}{2\lambda}\left[-(\rho + 3p)\pi_{\mu\nu} + \pi_{\alpha\langle\mu}\pi_{\nu\rangle}^{\alpha}\right] + \frac{6}{\kappa^2\lambda}\mathcal{P}_{\mu\nu}, \quad (3)$$

Raychaudhuri's equation ( $\Theta = 3H$ ),

$$\dot{\Theta} + \frac{1}{3}\Theta^2 + \sigma^{\mu\nu}\sigma_{\mu\nu} + \frac{1}{2}\kappa^2(\rho + 3p) - \Lambda = -\frac{1}{2\lambda\kappa^2}\left[\kappa^4(2\rho^2 + 3\rho p) + 12\mathcal{U}\right], \quad (4)$$

the dark energy propagation equation,

$$\dot{\mathcal{U}} + \frac{4}{3}\Theta\mathcal{U} + \sigma^{\mu\nu}\mathcal{P}_{\mu\nu} = \frac{\kappa^4}{12}\left[3\pi^{\mu\nu}\dot{\pi}_{\mu\nu} + 3(\rho + p)\sigma^{\mu\nu}\pi_{\mu\nu} + \Theta\pi^{\mu\nu}\pi_{\mu\nu} - \sigma^{\mu\nu}\pi_{\alpha\mu}\pi_{\nu}^{\alpha}\right]. \quad (5)$$

Here,  $H$  is the Hubble parameter;  $\sigma_{\mu\nu}$  is the shear tensor;  ${}^{(3)}\mathcal{R}_{\mu\nu}$  is the 3-curvature;  $\rho$  the energy-density;  $p$  the isotropic pressure;  $\pi_{\mu\nu}$  the

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<sup>1</sup> This would have some interesting consequences for the Weyl curvature conjecture [22–25].

anisotropic stress tensor;  $\mathcal{U}$  is the nonlocal dark energy; and  $\mathcal{P}_{\mu\nu}$  is the nonlocal bulk graviton stress tensor.

Note that there are no propagation equations for the nonlocal bulk graviton stress tensor,  $\mathcal{P}_{\mu\nu}$  (see also [28]).

The effect from the anisotropic stresses on the isotropic initial singularity can be seen by the following considerations (following [29, 30]). We assume that we are close to a FRW universe, and hence  $H, \rho \propto t^{-1}$  (note that in the flat isotropic limit we have  $H^2 \propto \rho^2$  for brane-world models). For simplicity, we further assume that the stress tensor is on the general form<sup>2</sup>  $\pi_{\mu\nu} = C_{\mu\nu}\rho_r$  where  $C_{\mu\nu}$  is a constant trace-free matrix, and  $\rho_r$  is the energy-density of a radiation type of fluid causing the anisotropic stresses (hence, we assume that the isotropic pressure  $p_r$  is  $p_r = \rho_r/3$ ). Including also an isotropic fluid with equation of state  $p_i = (\gamma - 1)\rho_i$ , we note that the isotropic past singularity is unstable due to the anisotropic stresses whenever  $\gamma \leq 4/3$ . Thus this simple investigation leads to the conclusion: *For brane-worlds with isotropic perfect fluids with  $\gamma \leq 4/3$  the initial singularity cannot be isotropic in the presence of stresses of type  $\pi_{\mu\nu} = C_{\mu\nu}\rho_r$ , and thus for magnetic fields in particular.* The anisotropic stresses will therefore be very important for the early time behaviour of brane-world models. The initial singularity is *matter dominated* in contrast to the shear dominated singularity in GR.

This makes us wonder whether the inclusion of a magnetic field (or more general types of stresses) will make the chaos come back to the Bianchi type I model. More generally we can ask ourself: What is the nature of the singularity for a magnetic brane-world?

To investigate this we used the dynamical systems approach and considered a Bianchi type I brane-world [31] with an isotropic perfect fluid,  $p = (\gamma - 1)\rho$ ; a magnetic field,  $\pi_{\mu\nu} = -B_\mu B_\nu + \frac{1}{3}B^2 h_{\mu\nu}$ ; and  $\mathcal{P}_{\mu\nu} = 0$  in order for the equations to close.

By finding all the equilibrium points and by investigating their stability, we found the following two (one for  $\gamma \leq 4/3$ ) **past attractors**:

1. Isotropic FRW:

Scale factor:  $a \propto t^{\frac{1}{3\gamma}}$

Past attractor for  $\gamma > \frac{4}{3}$ .

2. An anisotropic solution:

A past attractor with metric

$$ds^2 = -dt^2 + t^{-\frac{1}{68}(\sqrt{3345}-43)}dx^2 + t^{\frac{1}{2}}(dy^2 + dz^2). \quad (6)$$

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<sup>2</sup> This includes the magnetic field case.

The magnetic field and the dark-energy term diverge like  $B^2 \propto t^{-1}$  and  $\mathcal{U} \propto t^{-2}$ , respectively. This is an **attractor for all values of  $\gamma$** . For more details about the equilibrium points and their nature, consult [31].

The presence of an attractor solution in the magnetic Bianchi type I model may indicate that there are **no chaos** in the brane-world models.

We now see how the chaos is avoided in the brane-world model. The brane-world is very sensitive to the matter it contains which actually dictates the initial singularity almost entirely. The initial singularity does by no means need to be isotropic for these models, as a matter of fact, it can equally well be a matter/dark-energy dominated anisotropic singularity. The presence of the dark-energy term indicates that the Weyl tensor in the bulk is non-zero and could therefore be important initially. This may be a signal that the assumption that the nonlocal bulk graviton stress tensor is zero is an artificial and unnatural one. However, work done by others [32] indicates that in the absence of intrinsic stresses the past asymptotes does not change significantly with the inclusion of a non-zero  $\mathcal{P}_{\mu\nu}$ . This goes in favour of our assumption, but further work is needed on this issue.

#### 4. Conclusion

We have seen how anisotropic stresses can alter the behaviour at early times for brane-world models. At the high energies which occur at early times in the evolution of the universe, the brane-world is very dependent on the physical matter content. This is in contrast to the classical general relativistic case, where the early times is mostly dominated by the shear. In particular, we saw how the inclusion of anisotropic stresses could change the behaviour of the magnetic Bianchi type I model quite drastically; the classical model has a chaotic past, while the brane-world has not.

It should be noted that even though we have ruled out a chaotic behaviour in the past of the mixmaster type, we have not ruled out other types of chaos for brane-world models. For example, the chaos in models with Yang-Mills fields at late times [33], is not ruled out by our analysis. Further work is needed on this issue.

Notwithstanding that the brane-world model is very unlikely to have chaos in the past, the initial singularity could still be very anisotropic. This has later been suggested by others [34]. More specifically, the initial singularity in the magnetic Bianchi type I brane-world can only be isotropic if the universe contains an isotropic fluid stiffer than radiation (thus if  $\gamma > 4/3$ ).

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